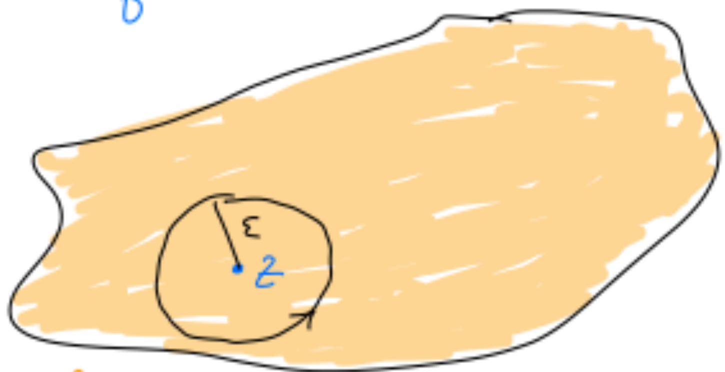
Recall: Mean value property of holomorphic fcns:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\, d\theta$$

Hypotheses:
f holomorphic on D,
$\overline{B(z,\varepsilon)} \subseteq D$.



Consequence of applying CIF to this specific circle around z.

Let's consider the abs value of $f(z)$:

$$\Rightarrow |f(z)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\, d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z+\varepsilon e^{i\theta})|\, |d\theta|$$

$$\leq \max_{\theta\in[0,2\pi]} |f(z+\varepsilon e^{i\theta})|$$

This is true for any ε as long as $\overline{B(z,\varepsilon)} \subseteq \text{Domain}(f)$.

max of $|f(z)|$ here $\geq |f(z)|$

Take limit as $\varepsilon \to$ (distance from z to boundary of domain)
if f is bounded, then you get a similar fact.

Maximum Modulus Principle (Version 1). Suppose f is holomorphic on a connected open set U, and $|f|$ achieves a maximum on $z_0 \in U$. Then f is is constant on U.

Proof:

① For any $B(z_0,\varepsilon) \subseteq U$ s.t. $\overline{B(z_0,\varepsilon)} \subseteq U$, we have from the calculation above.

$$|f(z_0)| \le \max_{z \in \partial B(z_0,\varepsilon)} |f(z)| \quad ; \forall \varepsilon \text{ with that property.}$$

From mean value property,

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\varepsilon e^{i\theta})|\,d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0)|}_{\text{because this is the max of } |f(z)| \text{ in } U}\,d\theta = |f(z_0)|$$

Then $$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\varepsilon e^{i\theta})|\,d\theta$$

$$\Rightarrow \left|f(z_0+\varepsilon e^{i\theta})\right| = |f(z_0)| \quad \forall \varepsilon, \theta$$

$\therefore$ $|f|$ is constant on that $\overline{B(z_0,\varepsilon)}$.
where ε is the largest possible.

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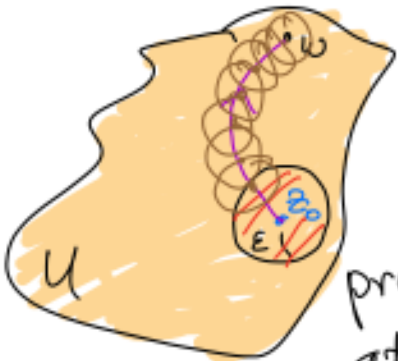
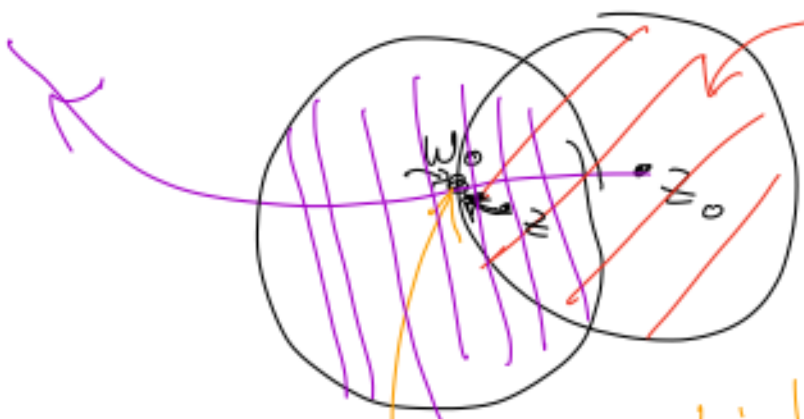
(2) for any other $w \in U$



cover a path from z_0 to w with disks that overlap. Using the previous argument, $|f(z)| = |f(z_0)|$ at all the values in each of those disks.



$|f(z_0)|$ constant by (1)

$$|f(w_0)| = \lim_{z\to w_0} |f(z)| \quad \text{since } f \text{ is continuous}$$

$$= \lim_{z\to w_0} |f(z_0)| = |f(z_0)|$$

$\Rightarrow$ w_0 is a max. for $|f(z)|$ again $\Rightarrow$ $|f(z)|$. $\Rightarrow$ use (1) $|f(z)|$ is constant on the $2^{\underline{nd}}$ disk.

Thus $|f(w)| = |f(z_0)|$. Thus $|f(z)|$ is constant on all of U.

Why does $|f(z)|$ constant imply that $f(z)$ is constant?

Let $f(z) = f(x+iy) = u(x,y) + i(v(x,y))$ for $x+iy \in U$, $x,y \in \mathbb{R}$, $u(x,y), v(x,y) \in \mathbb{R}$.

$|f(z)|^2 = u^2 + v^2$. Suppose $|f(z)|$ is constant.

$$\Rightarrow \left(|f(z)|^2\right)_x = 0 = 2uu_x + 2vv_x$$

$$\Rightarrow \left(|f(z)|^2\right)_y = 0 = 2uu_y + 2vv_y = 0$$

We have $u_x = v_y$, $u_y = -v_x$

$$\Rightarrow 2uv_y + 2vv_x = 0 \quad (1)$$

$$-2uv_x + 2vv_y = 0 \quad (2)$$

$(2)\cdot u \Rightarrow -2u^2v_x + 2uvv_y = 0$

$(1)\cdot v \Rightarrow 2uvv_y + 2v^2v_x = 0$

$(1)\cdot v - (2)u \Rightarrow 2(v^2+u^2)v_x = 0$

$$\Rightarrow 2|f(z)|^2 v_x = 0$$

$\Rightarrow |f(z)|^2 = 0$ or $V_x = 0 \Rightarrow -u_y = 0$

Similarly $|f(z)|^2 = 0$ or $V_y = 0 \Rightarrow u_x = 0$

$\Rightarrow f(z) = 0$ or V is constant.

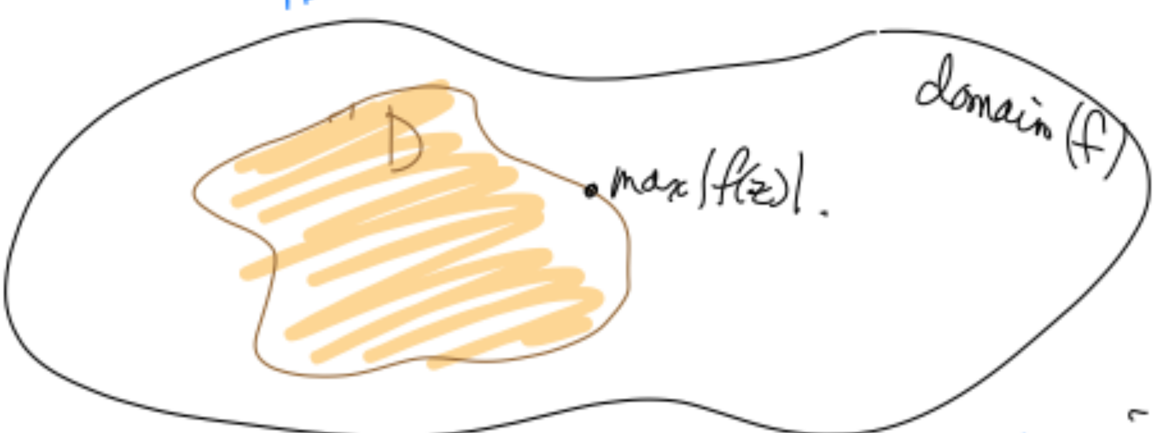
and u is constant

$\Rightarrow f(z)$ is constant on U.

• Corollary (Version 2). If D is a bounded domain such that for a holom fcn f with $\overline{D} \subseteq \text{domain}(f)$, then

$$\max_{z \in \overline{D}} |f(z)| = \max_{z \in \partial D} |f(z)|.$$

ie $|f|\big|_{\overline{D}}$ achieves its max on ∂D.



Proof: If $|f|$ achieves its max on $z_0 \in D$, then by version 1, $f(z)$ is const. on $D \Rightarrow |f(z)|$ is also constant on $D \Rightarrow |f|\big|_{\overline{D}}$ achieves its max on ∂D.. Otherwise, $|f|\big|_{\overline{D}}$ must achieve its max on ∂D.,

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<u>Note:</u> Since D is bounded, $\overline{D}$ is closed and bounded, and $|f|$ is continuous on $\overline{D}$, so by the extreme value theorem, $|f|$ must achieve its maximum and minimum on $\overline{D}$ somewhere. (either interior or boundary).

[Extreme Value Thm: A continuous, real-valued fcn on a compact set must achieve its max & min on that set.]

<u>Note:</u> There is a version of this for $(\min |f(z)|)$, but it only works if f is never zero in the domain D (or its closure). (ie. have to add that assumption) Without the extra assumption, $|f(z)|$ can be zero inside the domain and not

be constant.

Taylor series of holomorphic functions.

Our favorite:

$$f(z) = \frac{1}{1-z} = 1 + \sum_{k\geq 1} z^k \quad \left(\text{Geometric series}\right)$$

iff $|z|<1$. The Taylor series converges exactly to the function $\frac{1}{1-z}$ on the unit disk.

We can use this to find more general Taylor series:

eg.
$$\frac{2+i}{1-2i+z} = \frac{(2+i)}{(1-2i)} \frac{1}{\left(1+\frac{z}{1-2i}\right)}$$

$$= \frac{(2+i)}{(1-2i)} \frac{1}{1-\left(-\frac{z}{1-2i}\right)} = \frac{(2+i)}{(1-2i)}\left(\frac{1}{1-\left(\frac{z}{2i-1}\right)}\right)$$

$$= \frac{2+i}{1-2i} \sum_{k\geq 0} \left(\frac{z}{2i-1}\right)^k \quad \text{converges iff} \quad \left|\frac{z}{2i-1}\right| < 1$$

$\Longleftrightarrow \quad |z| < |2i-1| = \sqrt{5}$

$\therefore$ This Taylor series converges to the fcn.

Summary: Let $f(z) = \dfrac{2+i}{1-2i+z} = \sum_{k\geq 0} \underbrace{\left(\dfrac{2+i}{(1-2i)(2i-1)^k}\right)}_{a_k} z^k$,

(and it converges to $f(z)$
if $|z| < \sqrt{5}$.)

What if we want to expand around a different point?

Find out next time.